Exact solitary wave solutions for a discrete $\lambda\phi^4$ field theory in 1+1 dimensions

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We have found exact, periodic, time-dependent solitary wave solutions of a discrete ϕ^4 field theory model. For finite lattices, depending on whether one is considering a repulsive or attractive case, the solutions are Jacobi elliptic functions, either $\mathrm{sn}(x,m)$ [which reduce to the kink function $\tanh(x)$ for $m\to 1$], or they are $\mathrm{dn}(x,m)$ and $\mathrm{cn}(x,m)$ [which reduce to the pulse function $\mathrm{sech}(x)$ for $m\to 1$]. We have studied the stability of these solutions numerically, and we find that our solutions are linearly stable in most cases. We show that this model is a Hamiltonian system, and that the effective Peierls-Nabarro barrier due to discreteness is zero not only for the two localized modes but even for all three periodic solutions. We also present results of numerical simulations of scattering of kink-antikink and pulse-antipulse solitary wave solutions.

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I. INTRODUCTION

Discrete nonlinear equations are ubiquitous and play an important role in diverse physical contexts [1,2]. Some examples of integrable discrete equations include the Ablowitz-Ladik (AL) lattice [3] and the Toda lattice [4]. Certain non-Abelian discrete integrable models are also known [5]. There are many nonintegrable discrete nonlinear equations such as the discrete sine-Gordon (DSG) [6,7], the discrete nonlinear Schrödinger (DNLS) equation [8], and the Fermi-Pasta-Ulam (FPU) problem [9]. DSG is a physical realization of the dynamics of dislocations in crystals where it is known as the Frenkel-Kontorova model [10]. It also arises in the context of ferromagnets with planar anisotropy [11], adsorption on a crystal lattice, and pinned charge-density waves [12]. Similarly, DNLS plays a role in the propagation of electromagnetic waves in doped glass fibers [13] and other optical waveguides [14], and it describes Bose-Einstein condensates in optical lattices [15]. FPU has served as a fertile paradigm for understanding solitons, discrete breathers, intrinsic localized modes, chaos, anomalous transport in low-dimensional systems, and the fundamentals of statistical mechanics [9]. A discrete double well or discrete ϕ^4 equation is a model for structural phase transitions [16], and may be relevant for a better understanding of the collisions of relativistic kinks [17].

Obtaining exact (solitonlike) solutions is always desirable, particularly for discrete systems where the notion of a discreteness (or Peierls-Nabarro) barrier [18,19] is an important one related to the pinning of dislocations [10]. In addition, exact solutions allow one to calculate certain important physical quantities analytically as well as serving as diagnostics for simulations. Discrete models generally break translational invariance leading to a barrier for dislocation motion which occurs in the discrete sine-Gordon [20] or Frenkel-Kontorova [21,22] models. However, starting with a generalization of the nonlinear Schrödinger equation, it is possible to obtain static solutions of an AL-type discretization of the ϕ^4 model without a Peierls-Nabarro barrier [23]. Recently derived summation identities [24] involving Jacobi elliptic functions [25,26] led to exact periodic solutions of a modified DNLS equation [27]. In this paper, we exploit similar identities to obtain exact time-dependent periodic solutions of the discrete ϕ^4 model.

The paper is organized as follows. In the next section, we summarize the exact solitary wave solutions of the continuum ϕ^4 model. In Sec. III by identifying the relevant elliptic function identities, we derive a discrete ϕ^4 model which allows for exact static solutions. We then obtain these solutions and study their stability. In Sec. IV, we explicitly write down the corresponding Hamiltonian dynamics using a modified Poisson bracket algebra and obtain moving kink and pulse solutions. Section V contains numerical results for the scattering of both kink- and pulselike solitary waves. Kink-antikink collisions appear to create a breather with some radiation, and pulse-antipulse collisions lead to a flip and little radiation. In Sec. VI we compute the energy of the solitary waves, and show that the Peierls-Nabarro barrier [18,19] due to discreteness is zero not only for the two localized solutions but even for the three periodic solutions. Finally, we summarize our main findings in Sec. VII.

II. CONTINUUM SOLITARY WAVES

The double-well potential with the coupling parameter λ and the two minima at $\phi=\pm a$,

$$V=\frac{\lambda}{4}(\phi^2-a^2)^2, \tag{2.1}$$

leads to the following relativistic field equation:

$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}-\lambda\phi(\phi^2-a^2)=0. \tag{2.2}$$

For $\lambda>0$, if one assumes a moving periodic solution with velocity v in terms of the Jacobi elliptic function $\mathrm{sn}(\mathrm{x},\mathrm{m})$ with modulus m,

$$\phi(x,t) = Aa\,\mathrm{sn}[\beta(x + x_0 - vt), m], \quad (2.3)$$

one finds that

$$\beta = \left[\frac{\lambda a^2}{(1 - v^2)(1 + m)}\right]^{1/2},$$

$$A = \sqrt{\frac{2m}{1 + m}}. \quad (2.4)$$

The usual kink solitary wave is obtained in the limit $m \to 1$, as

$$\phi(x,t) = a \tanh\left\{\left[\frac{\lambda a^2}{2(1 - v^2)}\right]^{1/2}(x + x_0 - vt)\right\}. \quad (2.5)$$

If instead $\lambda < 0$, then there are dn Jacobi elliptic function solutions to the field equations. Assuming

$$\phi(x,t) = Aa\,\mathrm{dn}[\beta(x + x_0 - vt), m], \quad (2.6)$$

we find a moving periodic pulse solution with

$$\beta = \left[\frac{-\lambda a^2}{(1 - v^2)(2 - m)}\right]^{1/2},$$

$$A = \sqrt{\frac{2}{2 - m}}. \quad (2.7)$$

In fact, there is another pulse solution in terms of the Jacobi elliptic function of cn type. Assuming

$$\phi(x,t) = Aa\,\mathrm{cn}[\beta(x + x_0 - vt), m], \quad (2.8)$$

we obtain a moving periodic solution with

$$\beta = \left[\frac{-\lambda a^2}{(1 - v^2)(2m - 1)}\right]^{1/2},$$

$$A = \sqrt{\frac{2m}{2m - 1}}. \quad (2.9)$$

Note that this solution is valid only if $m > 1/2$.

The usual pulse solitary wave is obtained in the limit $m \to 1$, as

$$\phi(x,t) = a\sqrt{2}\,\mathrm{sech}\left[\left(\frac{-\lambda a^2}{1 - v^2}\right)^{1/2}(x + x_0 - vt)\right]. \quad (2.10)$$

Note that similar solutions and their stability were considered by Aubry in the context of structural phase transitions [28]. In the rest of this paper, we will refer to the sn solutions as kinklike, and to the dn and cn solutions as pulselike.

III. DISCRETIZATION OF $\lambda\phi^4$ FIELD THEORY

A naive discretization of the field equation above is

$$-\ddot{\phi}_n + \frac{1}{\epsilon^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n(a^2 - \phi_n^2) = 0, \quad (3.1)$$

where ϵ is the lattice parameter and the overdots represent time derivatives. However, this equation does not admit solutions of the form

$$\phi_n(t) = Aa\,\mathrm{sn}(\beta[(n + c)\epsilon - vt], m), \quad (3.2)$$

where c is an arbitrary constant. To find solutions, one has to modify the naive discretization. The key for understanding how to modify the equation comes from the following identities [24] of the Jacobi elliptic functions.

(i) sn:

$$m\,\mathrm{sn}(x,m)^2[\mathrm{sn}(x + \beta,m) + \mathrm{sn}(x - \beta,m)] = \mathrm{ns}^2(\beta,m)[\mathrm{sn}(x + \beta,m) + \mathrm{sn}(x - \beta,m)] - 2\mathrm{cs}(\beta,m)\mathrm{ds}(\beta,m)\mathrm{sn}(x,m). \quad (3.3)$$

(ii) cn:

$$m\,\mathrm{cn}(x,m)^2[\mathrm{cn}(x + \beta,m) + \mathrm{cn}(x - \beta,m)] = -\mathrm{ds}^2(\beta,m)[\mathrm{cn}(x + \beta,m) + \mathrm{cn}(x - \beta,m)] + 2\mathrm{cs}(\beta,m)\mathrm{ns}(\beta,m)\mathrm{cn}(x,m). \quad (3.4)$$

(iii) dn:

$$\mathrm{dn}(x,m)^2[\mathrm{dn}(x + \beta,m) + \mathrm{dn}(x - \beta,m)] = -\mathrm{cs}^2(\beta,m)[\mathrm{dn}(x + \beta,m) + \mathrm{dn}(x - \beta,m)] + 2\mathrm{ds}(\beta,m)\mathrm{ns}(\beta,m)\mathrm{dn}(x,m), \quad (3.5)$$

where

$$\mathrm{ns}(x,m) = \frac{1}{\mathrm{sn}(x,m)}, \quad \mathrm{cs}(x,m) = \frac{\mathrm{cn}(x,m)}{\mathrm{sn}(x,m)}, \quad (3.6)$$

$$\mathrm{ds}(x,m) = \frac{\mathrm{dn}(x,m)}{\mathrm{sn}(x,m)}.$$

For the sake of brevity, in what follows we will suppress the modulus m in the argument of the Jacobi elliptic functions, except when needed for added clarity.

A. Static lattice solutions

Consider the static lattice equation (3.1),

$$\frac{1}{\epsilon^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n(a^2 - \phi_n^2) = 0. \quad (3.7)$$

Here $\phi_n \equiv \phi[\beta\epsilon(n+c), m]$. A general ansatz to obtain the solution is to multiply the second difference operator by the factor $(1 - \alpha\phi_n^2)$, with α chosen so that we get a consistent set of equations. That is, we consider instead the equation

$$\frac{1 - \alpha\phi_n^2}{\epsilon^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n(a^2 - \phi_n^2) = 0. \quad (3.8)$$

Using α to eliminate the ϕ_n^3 term leads to the result

$$\alpha = \lambda\epsilon^2/2. \quad (3.9)$$

This implies, in the static case, that the lattice equation is just a smeared discretization of the ϕ_n^3 term. Namely, one merely needs to study the lattice equation,

$$\frac{1}{\epsilon^2}(\phi_{n+1}+\phi_{n-1}-2\phi_n)=\frac{\lambda\phi_n^2}{2}(\phi_{n+1}+\phi_{n-1})-\lambda\phi_n a^2. \tag{3.10}$$

Assume a solution of the form (for $\lambda > 0$)

$$\phi_n = Aa\ \mathrm{sn}_n, \tag{3.11}$$

where sn_n denotes $\mathrm{sn}[\beta\epsilon(n+c),m]$ with c being an arbitrary constant. Note that we only need to consider c between 0 and $\frac{1}{2}$ (half the lattice spacing). On using the identity (3.3), and matching the coefficients of sn_n and $\mathrm{sn}_{n+1}+\mathrm{sn}_{n-1}$, we obtain

$$A^2a^2 = \frac{2m\ \mathrm{sn}^2(\beta\epsilon)}{\lambda\epsilon^2} \tag{3.12}$$

and

$$\lambda a^2 = \frac{2}{\epsilon^2}[1-\mathrm{dn}(\beta\epsilon)\mathrm{cn}(\beta\epsilon)]. \tag{3.13}$$

For $(\epsilon\to 0)$ we obtain our continuum result (2.4). Also as $m\to 1$ we recover the usual kink solution

$$\phi_n = a\tanh[\beta\epsilon(n+c)]. \tag{3.14}$$

If instead, we have $\lambda < 0$ and we assume

$$\phi_n = Aa\ \mathrm{dn}_n, \tag{3.15}$$

we obtain

$$A^2a^2 = -\frac{2}{\lambda\epsilon^2}\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)} \tag{3.16}$$

and

$$\lambda a^2 = \frac{2}{\epsilon^2}\left[1-\frac{\mathrm{dn}(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)}\right]. \tag{3.17}$$

In the limit when the lattice spacing ϵ goes to zero, we recover the continuum result (2.7).

For $\lambda < 0$, we also have a possible solution of the form

$$\phi_n = Aa\ \mathrm{cn}_n. \tag{3.18}$$

We obtain

$$A^2a^2 = -\frac{2m}{\lambda\epsilon^2}\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{dn}^2(\beta\epsilon)} \tag{3.19}$$

and

$$\lambda a^2 = \frac{2}{\epsilon^2}\left[1-\frac{\mathrm{cn}(\beta\epsilon)}{\mathrm{dn}^2(\beta\epsilon)}\right]. \tag{3.20}$$

Again taking the lattice spacing to zero, we obtain the continuum case (2.9). Note that as before, this solution is only valid if $m > \frac{1}{2}$. For both these solutions, as $m\to 1$, we get the pulse solution

$$\phi_n = A\ a\ \mathrm{sech}[\beta\epsilon(n+c)]. \tag{3.21}$$

where $A=\sqrt{2}\cosh[\beta\epsilon/2]$.

B. Stability of solutions

Let us now discuss the stability of these static solutions. To that purpose, let us expand

$$\phi_n = \phi_n^{(s)} + \psi_n \exp(-i\omega t),$$

where $\phi_n^{(s)}$ is the known solitary wave exact solution and ψ_n is a small perturbation. Then we find that to the lowest order in ψ_n the stability equation is

$$\omega^2\psi_n + \frac{1}{\epsilon^2}(\psi_{n+1}+\psi_{n-1}-2\psi_n) - \lambda\psi_n\phi_n^{(s)}[\phi_{n+1}^{(s)}+\phi_{n-1}^{(s)}] + \lambda\psi_n a^2 - \frac{\lambda\phi_n^2}{2}(\psi_{n+1}+\psi_{n-1}) = 0. \tag{3.22}$$

Using the identities (3.3)–(3.5), the combination $[\phi_{n+1}^{(s)}+\phi_{n-1}^{(s)}]$ can be written as a function of ϕ_n alone. Thus schematically we have

$$(\omega^2 - f_n)\psi_n + g_n(\psi_{n+1}+\psi_{n-1}) = 0 \tag{3.23}$$

or

$$\psi_{n+1}+\psi_{n-1}+h_n\psi_n = 0, \tag{3.24}$$

with

$$h_n = \frac{\omega^2 - f_n}{g_n}, \tag{3.25}$$

where f_n and g_n are well defined functions. We also have that the solutions are periodic on the lattice with N sites $(\psi_{n+N}=\psi_n)$. If we write the system of N equations in matrix form as

$$\mathbf{A}[\psi] = \omega^2[\psi], \tag{3.26}$$

then the condition for nontrivial solutions is that

$$\det|\mathbf{A} - \omega^2\mathbf{1}| = 0. \tag{3.27}$$

In our simulations, we require that the solution has exactly one period in the model space. Therefore, the equations for β and A are supplemented by the equations

$$N\beta\epsilon = 4K(m) \tag{3.28}$$

for solutions $\phi_n=\mathrm{sn}(\beta n\epsilon)$ or $\phi_n=\mathrm{cn}(\beta n\epsilon)$, and

$$N\beta\epsilon = 2K(m) \tag{3.29}$$

when $\phi_n=\mathrm{dn}(\beta n\epsilon)$. Here $K(m)$ is the complete elliptic integral of the first kind [25,26]. Hence, once the parameters m, λ, and a are specified, we need to solve a system of equations for β, ϵ, and A, i.e., we have to solve Eqs. (3.12), (3.13), and (3.28) or Eqs. (3.16), (3.17), and (3.29), or Eqs. (3.19), (3.20), and (3.28), for the solutions sn, dn, and cn, respectively, for β, ϵ, and A as a function of N.

Typical results are depicted in Figs. 1 and 2 for kinklike sn solutions and in Figs. 3 and 4, for pulselike dn solutions, respectively. Similar results can also be obtained in the case of the cn-type pulse solution. Given the requirement of periodic boundary conditions, the functional form of the kinklike and pulselike solutions, respectively, gives rise to a minimum number of grid points necessary to render the discretization

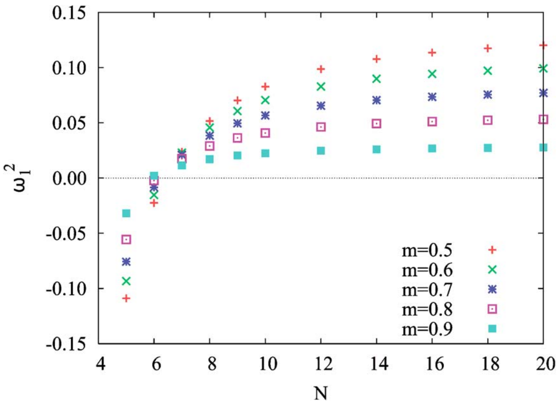

FIG. 1. (Color online) Kinklike ($\lambda > 0$) sn case: N dependence of the lowest eigenvalue, ω_1^2, for $\lambda = 1$, $a = 1$, and various values of the elliptic modulus m.

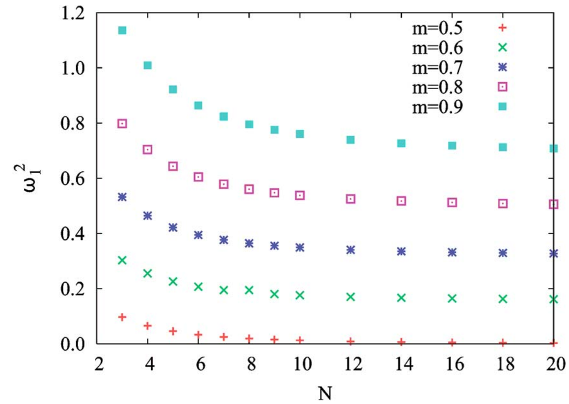

FIG. 3. (Color online) Pulselike ($\lambda < 0$) dn case: N dependence of the lowest eigenvalue, ω_1^2, for $\lambda = -1$, $a = 1.5$, and various values of m.

mathematically consistent. Therefore, for the case of the kinklike solutions we find from Fig. 1 that stability requires $N > 7$ lattice sites, while for pulselike solutions we have $N > 2$. In the pulselike case, we find stability for arbitrary values of N. The magnitude of the lowest eigenvalues increases with N in the kinklike case, while it decreases with N for pulselike solutions. For fixed m, λ, and A, the lattice spacing, ϵ, is not an independent quantity, but it is a well-defined function of N. As seen from Figs. 2 and 4, the lattice spacing, ϵ, is always a decreasing function of N.

IV. HAMILTONIAN DYNAMICS

In this section, we demonstrate that our discrete model is a Hamiltonian system. The equation which the static solutions obey, Eq. (3.8), can be written as

$$\frac{1}{\epsilon^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda \frac{\phi_n(a^2 - \phi_n^2)}{1 - \alpha\phi_n^2} = 0. \tag{4.1}$$

We recognize that this equation can be derived from the potential function

$$V_0[\phi_n] = \sum_n \frac{(\phi_{n+1} - \phi_n)^2}{2\epsilon^2} + \lambda \sum_n \int d\phi_n \frac{\phi_n(a^2 - \phi_n^2)}{1 - \alpha\phi_n^2}. \tag{4.2}$$

Performing the integral, we obtain explicitly

$$V_0[\phi_n] = \sum_n \frac{(\phi_{n+1} - \phi_n)^2}{2\epsilon^2} - \lambda \frac{\phi_n^2}{2\alpha} + \lambda(a^2\alpha - 1)\frac{\ln(1 - \alpha\phi_n^2)}{2\alpha^2}. \tag{4.3}$$

One easily verifies that

$$-(1 - \alpha\phi_n^2)\frac{\partial V_0}{\partial \phi_n} = \frac{1 - \alpha\phi_n^2}{\epsilon^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n(a^2 - \phi_n^2). \tag{4.4}$$

The continuum limit $\alpha \to 0$ (or $\epsilon \to 0$) of the last two terms in Eq. (4.3) is given by

$$\frac{\lambda}{4}(\phi_n^2 - a^2)^2 - \frac{\lambda}{4}a^4. \tag{4.5}$$

We will now show that if we want our class of solutions to be the static limit of a Hamiltonian dynamical system, we

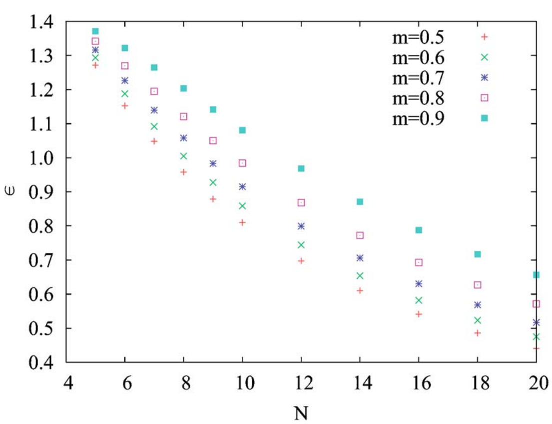

FIG. 2. (Color online) Kinklike ($\lambda > 0$) sn case: N dependence of the lattice spacing, ϵ, for $\lambda = 1$, $a = 1$, and various values of m.

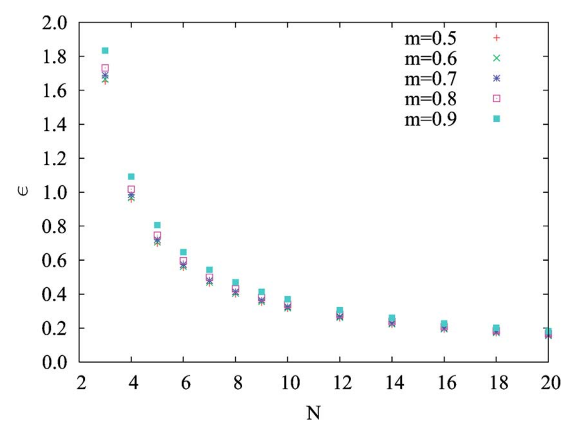

FIG. 4. (Color online) Pulselike ($\lambda < 0$) dn case: N dependence of the lattice spacing, ϵ, for $\lambda = -1$, $a = 1.5$, and various values of m.

will be able to have single solitary waves obey a simple equation, but general solutions will obey a more complicated dynamics with terms proportional to $\dot{\phi}^2$. For simplicity, we will assume that the Hamiltonian takes the form

$$H = \sum_n \left(\frac{\pi_n^2}{2} g[\phi_n] + V[\phi_n] \right), \tag{4.6}$$

with V given by V_0 in Eq. (4.3) plus possibly some additional terms that vanish in the continuum limit. Here π_n is the conjugate momentum and $g[\phi_n]$ a weight function. For generality we will assume, as in the case of the discrete DNLS equation [23], that an extended Poisson bracket structure exists [29,30], namely

$$\{\phi_m, \pi_n\} = \delta_{nm} f[\phi_n] \tag{4.7}$$

and

$$\dot{\phi}_n = \{\phi_n, H\} = \frac{\partial H}{\partial \pi_m}\{\phi_n, \pi_m\} = f[\phi_n]\frac{\partial H}{\partial \pi_n},$$

$$\dot{\pi}_n = \{\pi_n, H\} = \frac{\partial H}{\partial \phi_m}\{\pi_n, \phi_m\} = -f[\phi_n]\frac{\partial H}{\partial \phi_n}. \tag{4.8}$$

From our ansatz, Eq. (4.6), we obtain the first-order equations

$$\dot{\phi}_n = \pi_n f[\phi_n] g[\phi_n] \equiv \pi_n h[\phi_n] \tag{4.9}$$

and

$$\dot{\pi}_n = -f[\phi_n]\left(\frac{\pi_n^2}{2}\frac{\partial g}{\partial \phi_n} + \frac{\partial V}{\partial \phi_n} \right). \tag{4.10}$$

This leads to the following second-order differential equation for ϕ_n:

$$\ddot{\phi}_n = -f[\phi_n]h[\phi_n]\frac{\partial V}{\partial \phi_n} + \dot{\phi}^2\left(\frac{1}{h}\frac{\partial h}{\partial \phi_n} - \frac{f}{2h}\frac{\partial g}{\partial \phi_n} \right). \tag{4.11}$$

For this equation to have the previously found static solitary waves, as well as the correct continuum limit, we need only that

$$f[\phi_n]h[\phi_n] = f^2[\phi_n]g[\phi_n] = 1 - \alpha\phi_n^2. \tag{4.12}$$

Three convenient choices which lead to the same second-order equation of motion for ϕ_n are (a) $f=1$ (ordinary Poisson brackets), (b) $h=1$ (extended Poisson brackets), and (c) $g=1$ (extended Poisson brackets and conventional kinetic energy). First consider the case $f=1$. This requires

$$h[\phi_n] = g[\phi_n] = 1 - \alpha\phi_n^2. \tag{4.13}$$

For the case $h=1$, this leads to

$$f = 1 - \alpha\phi_n^2, \quad g = (1 - \alpha\phi_n^2)^{-1}. \tag{4.14}$$

For the case $g=1$, we obtain

$$f = h = \sqrt{1 - \alpha\phi_n^2}. \tag{4.15}$$

From all choices satisfying Eq. (4.12) we get, if we choose $V=V_0$ with V_0 given by Eq. (4.3), the equation of motion

$$\ddot{\phi}_n = \frac{1 - \alpha\phi_n^2}{\epsilon^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n(a^2 - \phi_n^2) - \frac{\alpha\dot{\phi}_n^2\phi_n}{1 - \alpha\phi_n^2}. \tag{4.16}$$

For $\alpha=\lambda\epsilon^2/2$ this equation has the previously found static lattice solitary wave solutions as well as the correct continuum limit. The third choice ($g=1$) has the property that the kinetic term in the Hamiltonian is the standard one, and all the dependence of the Hamiltonian on the parameter α is contained in the potential $V[\phi_n]$.

Unfortunately, the presence of the $\dot{\phi}^2$ term does not allow for single elliptic time-dependent solitary wave solutions that are of the form $\mathrm{sn}[\beta(x-ct)]$, $\mathrm{cn}[\beta(x-ct)]$, or $\mathrm{dn}[\beta(x-ct)]$ where $x=n\epsilon$. This is because the second derivative of the sn and dn (or cn) functions contains both linear as well as cubic terms, and the quantity $\dot{\phi}^2$ is a quartic polynomial in sn or dn (or cn). Thus the last term is equivalent to a nonpolynomial potential when applied to a single elliptic function solution. Therefore, in order to obtain a simple elliptic function solution in the time-dependent case, one must add a nonpolynomial potential to V_0 which is chosen to exactly cancel the last term when evaluated for a single elliptic solitary wave solution of the form $\mathrm{sn}[\beta(x-ct)]$, $\mathrm{cn}[\beta(x-ct)]$, or $\mathrm{dn}[\beta(x-ct)]$. This means that to obtain an equation that is derivable from a Lagrangian or a Hamiltonian, we should add to the static potential V_0 an additional contribution ΔV such that

$$(1 - \alpha\phi_n^2)\frac{\partial \Delta V}{\partial \phi_n} = -\frac{\alpha\phi_n\dot{\phi}_n^2}{1 - \alpha\phi_n^2}, \tag{4.17}$$

where ϕ_n is a single solitary wave described by a time-translated elliptic function. It will turn out that ΔV needed to obtain a simple solution of the elliptic kind is a velocity-dependent potential, which also depends on the type of solution (pulse- or kinklike).

In general, we have that the Hamiltonian dynamics leads to

$$\ddot{\phi}_n = -fh\frac{\partial V_0 + \Delta V}{\partial \phi_n} + \dot{\phi}_n^2\left(\frac{1}{h}\frac{\partial h}{\partial \phi_n} - \frac{f}{2h}\frac{\partial g}{\partial \phi_n} \right). \tag{4.18}$$

Since $fh=1-\alpha\phi_n^2$ and $h=fg$, it follows that

$$\frac{1}{h}\frac{\partial h}{\partial \phi_n} - \frac{f}{2h}\frac{\partial g}{\partial \phi_n} = -\frac{\alpha\phi_n}{1 - \alpha\phi_n^2}, \tag{4.19}$$

from where the above condition for ΔV follows readily. For solitary wave solutions of the sn, cn, and dn type, one wants to choose

$$\frac{\partial \Delta V}{\partial \phi_n} = \frac{a_1\phi_n^5 + a_2\phi_n^3 + a_3\phi_n}{(1 - \alpha\phi_n^2)^2}, \tag{4.20}$$

with a_i dependent on the choice of elliptic function, in order to cancel the effect of the $\dot{\phi}^2$ terms in the corresponding equation of motion.

Integrating, we get for ΔV

$$\Delta V = \frac{a_1\phi^2}{2\alpha^2} + \frac{1}{2(1-\alpha\phi^2)}\left(\frac{a_1}{\alpha^3} + \frac{a_2}{\alpha^2} + \frac{a_3}{\alpha}\right) + \left(\frac{a_1}{\alpha^3} + \frac{a_2}{2\alpha^2}\right)\ln(1-\alpha\phi^2). \quad (4.21)$$

If we assume a soliton solution of the form

$$\phi_n = Aa\,\mathrm{sn}\{\beta[(n+c)\epsilon - vt],m\}, \quad (4.22)$$

where c is an arbitrary constant, then

$$\dot{\phi}_n^2 = \beta^2 v^2 (Aa)^2 [1-(1+m)\mathrm{sn}^2 + m\,\mathrm{sn}^4], \quad (4.23)$$

which leads to

$$a_1 = -m\alpha\left(\frac{\beta v}{Aa}\right)^2, \quad (4.24)$$

$$a_2 = \alpha\beta^2 v^2(1+m), \quad (4.25)$$

$$a_3 = -\alpha(\beta v Aa)^2. \quad (4.26)$$

Explicitly, for the sn type of solitary wave (for $\lambda > 0$) we need to choose the extra potential term to satisfy

$$-\frac{\partial \Delta V}{\partial \phi_n} = \frac{\alpha(\beta v Aa)^2\phi_n}{(1-\alpha\phi_n^2)^2}\left[1 - \frac{(1+m)\phi_n^2}{(Aa)^2} + \frac{m\phi_n^4}{(Aa)^4}\right]. \quad (4.27)$$

If instead, we assume a pulse soliton solution (for $\lambda < 0$) of the form

$$\phi_n = Aa\,\mathrm{dn}\{\beta[(n+c)\epsilon - vt],m\}, \quad (4.28)$$

then we have

$$\dot{\phi}_n^2 = \beta^2 v^2 (Aa)^2[(m-1) + (2-m)\mathrm{dn}^2 - \mathrm{dn}^4], \quad (4.29)$$

which leads to the (different set) of coefficients $\{a_i\}$ for the extra term needed to be added to the Hamiltonian

$$a_1 = \alpha\left(\frac{\beta v}{Aa}\right)^2, \quad (4.30)$$

$$a_2 = -\alpha\beta^2 v^2(2-m), \quad (4.31)$$

$$a_3 = (1-m)\alpha(\beta v Aa)^2. \quad (4.32)$$

Explicitly, for the case of dn solitary waves, we choose our additional potential term to satisfy

$$-\frac{\partial \Delta V}{\partial \phi_n} = \frac{\alpha(\beta v Aa)^2\phi_n}{(1-\alpha\phi_n^2)^2}\left[(m-1) + \frac{(2-m)\phi_n^2}{(Aa)^2} - \frac{\phi_n^4}{(Aa)^4}\right]. \quad (4.33)$$

Finally, we assume a pulse solution (for $\lambda < 0$) of the form

$$\phi_n = Aa\,\mathrm{cn}\{\beta[(n+c)\epsilon - vt + c],m\}; \quad (4.34)$$

then we have

$$\dot{\phi}_n{}^2 = \beta^2 v^2 (Aa)^2[(1-m) + (2m-1)\mathrm{cn}^2 - m\,\mathrm{cn}^4], \quad (4.35)$$

which leads to the (different set) of coefficients $\{a_i\}$ for the extra term needed to be added to the Hamiltonian

$$a_1 = m\alpha\left(\frac{\beta v}{Aa}\right)^2, \quad (4.36)$$

$$a_2 = -\alpha\beta^2 v^2(2m-1), \quad (4.37)$$

$$a_3 = -(1-m)\alpha(\beta v Aa)^2. \quad (4.38)$$

Explicitly, for the case of cn solitary waves we choose our additional potential term to satisfy

$$-\frac{\partial \Delta V}{\partial \phi_n} = \frac{\alpha(\beta v Aa)^2\phi_n}{(1-\alpha\phi_n^2)^2}\left[(1-m) + \frac{(2m-1)\phi_n^2}{(Aa)^2} - \frac{m\phi_n^4}{(Aa)^4}\right]. \quad (4.39)$$

Because of our "fine tuning" of ΔV, in all cases, the solitary waves effectively obey the second-order differential difference equation,

$$-\ddot{\phi}_n + \frac{1-\alpha\phi_n^2}{\epsilon^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \lambda\phi_n(a^2 - \phi_n^2) = 0. \quad (4.40)$$

As long as α is proportional to ϵ^2, we expect that this equation has the correct continuum limit. Next, we demonstrate this for the three cases explicitly.

A. Positive λ and kink solutions

If we assume a solution of the form

$$\phi_n = Aa\,\mathrm{sn}\{\beta[(n+c)\epsilon - vt],m\}, \quad (4.41)$$

where c is an arbitrary constant, we obtain the equations

$$\mathrm{sn}^2(\beta\epsilon) = \alpha\frac{(Aa)^2}{m},$$

$$\lambda a^2 = \frac{2}{\epsilon^2}[1 - \mathrm{dn}(\beta\epsilon)\mathrm{cn}(\beta\epsilon)] - (1+m)\beta^2 v^2,$$

$$\lambda A^2 a^2 = 2m\frac{\mathrm{sn}^2(\beta\epsilon)}{\epsilon^2} - 2m\beta^2 v^2, \quad (4.42)$$

with the continuum limit ($\epsilon \to 0$) given by Eq. (2.4). From this we deduce that

$$A^2 = \frac{m[\mathrm{sn}^2(\beta\epsilon) - \beta^2 v^2\epsilon^2]}{[1 - \mathrm{dn}(\beta\epsilon)\mathrm{cn}(\beta\epsilon)] - (1+m)\epsilon^2\beta^2 v^2/2} \quad (4.43)$$

and

$$\frac{\alpha}{\lambda} = \frac{\mathrm{sn}^2(\beta\epsilon)}{2[\mathrm{sn}^2(\beta\epsilon)/\epsilon^2 - \beta^2 v^2]}. \quad (4.44)$$

For small lattice spacing ϵ we get the result

$$\alpha = \frac{\lambda \epsilon^2}{2(1-v^2)}. \qquad (4.45)$$

From Eq. (4.6) with the choice $g=1$, all the α dependence of the Hamiltonian is in $V[\phi_n]$, so the above result implies a velocity-dependent potential.

Localized mode. Let us now consider the $m \to 1$ limit in which case we get a localized kink solution

$$\phi_n = Aa \tanh[\beta(n+c)\epsilon - vt], \qquad (4.46)$$

where

$$A = 1, \quad \alpha a^2 = \tanh^2(\beta\epsilon),$$

$$\lambda a^2 = \frac{2}{\epsilon^2}\tanh^2(\beta\epsilon) - 2\beta^2 v^2. \qquad (4.47)$$

B. Negative λ and pulse solutions

If we assume a solution of the form

$$\phi_n = Aa\, \mathrm{dn}\{\beta([n+c]\epsilon - vt), m\}, \qquad (4.48)$$

where c is an arbitrary constant, we obtain the system of equations

$$\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)} = -\alpha(Aa)^2,$$

$$-\lambda a^2 = \frac{2}{\epsilon^2}\left[\frac{\mathrm{dn}(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)} - 1\right] - (2-m)\beta^2 v^2,$$

$$-\lambda A^2 a^2 = \frac{2}{\epsilon^2}\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)} - 2\beta^2 v^2, \qquad (4.49)$$

with the continuum limit ($\epsilon \to 0$) given by Eq. (2.7). From this we deduce that

$$A^2 = \frac{\mathrm{sn}^2(\beta\epsilon) - \beta^2 v^2 \epsilon^2 \mathrm{cn}^2(\beta\epsilon)}{[\mathrm{dn}(\beta\epsilon) - \mathrm{cn}^2(\beta\epsilon)] - (2-m)\epsilon^2\beta^2 v^2 \mathrm{cn}^2(\beta\epsilon)/2} \qquad (4.50)$$

and

$$\frac{\alpha}{\lambda} = \frac{\epsilon^2 \mathrm{sn}^2(\beta\epsilon)}{2[\mathrm{sn}^2(\beta\epsilon) - \epsilon^2\beta^2 v^2 \mathrm{cn}^2(\beta\epsilon)]}. \qquad (4.51)$$

For small ϵ we again have the relation (4.45). When $v \to 0$, we obtain $\alpha = \lambda\epsilon^2/2$. This exactly cancels the $\lambda\phi_n^3$ term in the equation of motion and we get the simple discretization for the time-independent case. In the time-dependent case, we again have the result that the potential needed is a function of the velocity of the solitary wave.

If instead we assume a solution of the form

$$\phi_n = Aa\, \mathrm{cn}\{\beta[(n+c)\epsilon - vt], m\}, \qquad (4.52)$$

where c is an arbitrary constant, we obtain the equations

$$m\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{dn}^2(\beta\epsilon)} = -\alpha(Aa)^2,$$

$$-\lambda a^2 = \frac{2}{\epsilon^2}\left[\frac{\mathrm{cn}(\beta\epsilon)}{\mathrm{dn}^2(\beta\epsilon)} - 1\right] - (2m-1)\beta^2 v^2,$$

$$-\lambda A^2 a^2 = \frac{2m}{\epsilon^2}\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{dn}^2(\beta\epsilon)} - 2m\beta^2 v^2, \qquad (4.53)$$

with the continuum limit ($\epsilon \to 0$) of Eq. (2.9). From this we deduce that

$$A^2 = \frac{m[\mathrm{sn}^2(\beta\epsilon) - \beta^2 v^2 \epsilon^2 \mathrm{dn}^2(\beta\epsilon)]}{[\mathrm{cn}(\beta\epsilon) - \mathrm{dn}^2(\beta\epsilon)] - (2m-1)\epsilon^2\beta^2 v^2 \mathrm{dn}^2(\beta\epsilon)/2} \qquad (4.54)$$

and

$$\frac{\alpha}{\lambda} = \frac{\epsilon^2 \mathrm{sn}^2(\beta\epsilon)}{2[\mathrm{sn}^2(\beta\epsilon) - \epsilon^2\beta^2 v^2 \mathrm{dn}^2(\beta\epsilon)]}. \qquad (4.55)$$

For small lattice spacing, ϵ, the parameters α and λ are related by Eq. (4.45).

Localized mode. In the limit of $m \to 1$, both cn and dn solutions reduce to the localized pulse solution

$$\phi_n = Aa\, \mathrm{sech}\{\beta[(n+c)\epsilon - vt]\}, \qquad (4.56)$$

where

$$-\alpha a^2 A^2 = \sinh^2(\beta\epsilon),$$

$$-\lambda a^2 = \frac{2}{\epsilon^2}[\cosh(\beta\epsilon) - 1] - \beta^2 v^2,$$

$$-\lambda a^2 A^2 = \frac{2}{\epsilon^2}\sinh^2(\beta\epsilon) - 2\beta^2 v^2. \qquad (4.57)$$

V. SCATTERING OF SOLITARY WAVES

When we have two solitary waves colliding with opposite velocity, then the equation of motion depends on whether we are considering sn, dn, or cn type solitary waves. In general, we have

$$\ddot{\phi}_n = \frac{1-\alpha\phi_n^2}{\epsilon^2}[\phi_{n+1} + \phi_{n-1} - 2\phi_n] + \lambda\phi_n(a^2 - \phi_n^2)$$
$$- \frac{\alpha}{1-\alpha\phi_n^2}\dot{\phi}_n{}^2\phi_n - (1-\alpha\phi_n^2)\frac{\partial \Delta V}{\partial \phi_n}, \qquad (5.1)$$

where the partial derivatives of ΔV are given by Eqs. (4.27), (4.33), and (4.39) for the three types of solutions, respectively. The ΔV term given by Eq. (4.21) adds an extra term to the energy-conservation equation. The conserved energy is given by

$$E = \sum_n \left[\frac{\dot{\phi}_n^2}{2(1-\alpha\phi_n^2)} + \frac{(\phi_{n+1} - \phi_n)^2}{2\epsilon^2} - \frac{\lambda}{2\alpha}\phi_n^2 + \lambda(a^2\alpha - 1)\frac{\ln(1-\alpha\phi_n^2)}{2\alpha^2}\right] + \Delta V. \qquad (5.2)$$

Typical scenarios are depicted in Figs. 5 and 6 for the scattering case of two kink-antikink waves and two pulse-

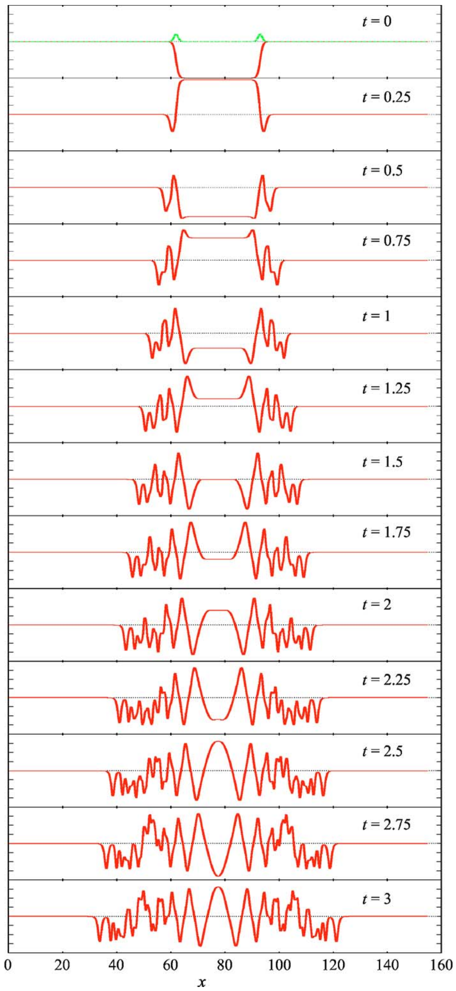

FIG. 5. (Color online) Scattering of single sn kink-antikink waves ($m=1$). For completeness, at $t=0$, we also depict the time derivative of the initial wave function.

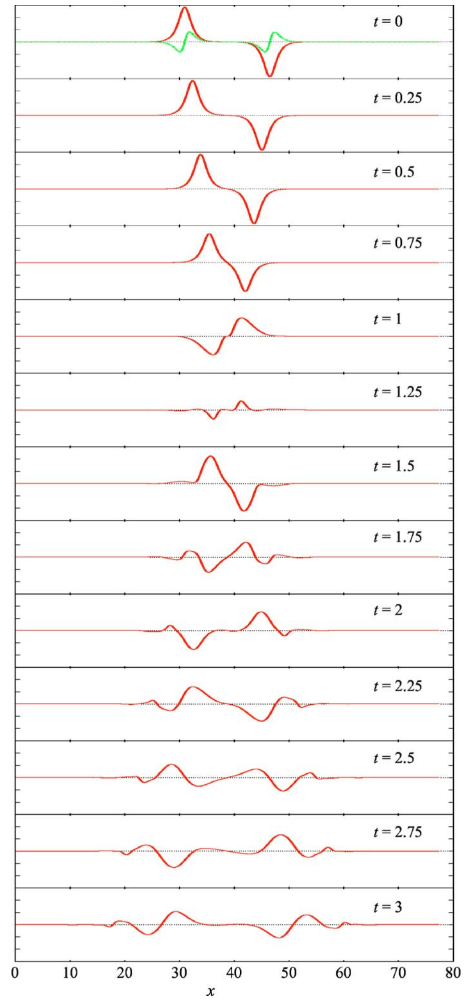

FIG. 6. (Color online) Scattering of single dn pulse-antipulse waves ($m=1$). For completeness, at $t=0$, we also depict the time derivative of the initial wave function.

antipulse waves, respectively. The solution of the kink-antikink "scattering" waves appears to correspond to a spatially localized, persistent time-periodic oscillatory bound state (or a breather [31,32]) with some radiation (phonons) at late times. Breathers are intrinsically dynamic nonlinear excitations and can be viewed as a bound state of phonons. The scattering of pulse-antipulse is different: there is a flip after collision and relatively less radiation. Here, our characterization of the dynamics for the two types of solutions is solely based on the visual analysis of the time evolution [33].

VI. ENERGY OF SOLITARY WAVES AND THE PEIERLS-NABARRO BARRIER

In a discrete lattice, there is an energy cost associated with moving a localized mode by a half lattice constant, known as Peierls-Nabarro (PN) barrier [18,19]. In the $m\to 1$ limit, the elliptic functions become localized and become either pulses or kinklike solitary waves. We shall now show that, rather remarkably, the PN barrier vanishes in this model for both types of localized modes. In fact, we prove an even stronger result, that the PN barrier is zero for all three periodic solutions (in terms of sn, cn, and dn).

We start from the conserved energy expression given by Eq. (5.2) with ΔV of Eq. (4.21). First of all, we shall show that in the time-dependent as well as in the static cases, the conserved energy is given by

$$E=\sum_n\left[-\frac{\phi_{n+1}\phi_n}{\epsilon^2}+B\ln(1-\alpha\phi_n^2)+A\right], \tag{6.1}$$

where the constants A and B vary for each case.

We first note that in view of Eqs. (4.17) and (4.20), we have

$$\frac{\dot{\phi}_n^2}{2(1-\alpha\phi_n^2)}=-\frac{1}{2\alpha(1-\alpha\phi_n^2)}(a_1\phi_n^4+a_2\phi_n^2+a_3). \tag{6.2}$$

Combining this term with ΔV of Eq. (4.21) in the energy expression (5.2), we then find that the conserved energy is given by

$$E=\sum_n\left[\left(\frac{1}{\epsilon^2}-\frac{\lambda}{2\alpha}+\frac{a_1}{\alpha^2}\right)\phi_n^2-\frac{\phi_{n+1}\phi_n}{\epsilon^2}+\frac{1}{2\alpha^2}\left(a_2+\frac{a_1}{\alpha}\right)+\left(\frac{\lambda a^2}{2\alpha}-\frac{\lambda}{2\alpha^2}+\frac{a_1}{\alpha^3}+\frac{a_2}{2\alpha^2}\right)\ln(1-\alpha\phi_n^2)\right]. \tag{6.3}$$

Quite remarkably we find that in the case of all the (i.e., sn, cn, as well as dn) solutions,

$$\frac{1}{\epsilon^2}-\frac{\lambda}{2\alpha}+\frac{a_1}{\alpha^2}=0, \tag{6.4}$$

where use has been made of relevant equations in Sec. IV. As a result, the ϕ_n^2 term vanishes. Thus the conserved energy takes a rather simple form as given by Eq. (6.1) in the case of all of our solutions, and the constants A and B are given by

$$A=\frac{1}{2\alpha^2}\left(a_2+\frac{a_1}{\alpha}\right), \tag{6.5}$$

$$B=\frac{\lambda a^2}{2\alpha}-\frac{\lambda}{2\alpha^2}+\frac{a_1}{\alpha^3}+\frac{a_2}{2\alpha^2}. \tag{6.6}$$

Here a_1,a_2,a_3 have different values for the three cases and are as given in Sec. IV. In the special case of the static solutions, there is a further simplification in that the constant term also vanishes, since $a_{1,2,3}$ are all zero in that case.

Let us now calculate the PN barrier in the case of the three periodic solutions obtained in Sec. IV and show that it vanishes in all three cases. Before we give the details, let us explain the key argument. If we look at the conserved energy expression as given by Eq. (6.1), we find that there are two c-dependent sums involved here. We also observe that in these expressions, time t and the constant c always come together in the combination

$$k_1=\beta(c\epsilon-vt). \tag{6.7}$$

Further, using the recently discovered identities for Jacobi elliptic functions, we explicitly show that for all the solutions the first sum is c-independent. Since the total energy E as given by Eq. (6.1) is conserved, its value must be independent of time t, and since time t and the constant c always come together in the combination k_1 as given by Eq. (6.7), it then follows that the second sum must also be c-independent and thus there is no PN barrier for any of our periodic, and hence also the localized pulse or kink, solutions.

In particular, the following three cyclic identities [24] will allow us to explicitly perform the first sum in Eq. (6.1):

$$m\,\mathrm{sn}(x)\mathrm{sn}(x+a)=-\,\mathrm{ns}(a)[Z(x+a)-Z(x)-Z(a)], \tag{6.8}$$

$$m\,\mathrm{cn}(x)\mathrm{cn}(x+a)=m\,\mathrm{cn}(a)+\mathrm{ds}(a)[Z(x+a)-Z(x)-Z(a)], \tag{6.9}$$

and

$$\mathrm{dn}(x)\mathrm{dn}(x+a)=\mathrm{dn}(a)+\mathrm{cs}(a)[Z(x+a)-Z(x)-Z(a)]. \tag{6.10}$$

Here $Z(x)\equiv Z(x,m)$ is the Jacobi zeta function [25,26]. In addition, we use the fact that

$$\sum_{n=1}^{N}\{Z[\beta\epsilon(n+1)+k_1,m]-Z(n\beta\epsilon+k_1,m)\}=0. \tag{6.11}$$

Let us now consider the sum as given by the first term of Eq. (6.1) for the three cases. We first consider the kinklike solution, which can also be written as

$$\phi_n=Aa\,\mathrm{sn}(n\beta\epsilon+k_1,m). \tag{6.12}$$

Using the identities given above, we obtain

$$\sum_{n=1}^{N}\left[-\frac{(\phi_{n+1}\phi_n)}{\epsilon^2}\right]=-\frac{(Aa)^2\mathrm{ns}(\beta\epsilon)}{m\epsilon^2}N\,Z(\beta\epsilon). \tag{6.13}$$

For the dn pulselike case, we have instead

$$\phi_n=Aa\,\mathrm{dn}(n\beta\epsilon+k_1,m), \tag{6.14}$$

and the first sum in Eq. (6.1) becomes

$$\sum_{n=1}^{N}\left[-\frac{\phi_{n+1}\phi_n}{\epsilon^2}\right]=-\frac{N(Aa)^2}{\epsilon^2}[\mathrm{dn}(\beta\epsilon)-\mathrm{cs}(\beta\epsilon)Z(\beta\epsilon)]. \tag{6.15}$$

Finally, for the cn pulselike case, we have

$$\phi_n=Aa\,\mathrm{cn}(n\beta\epsilon+k_1,m). \tag{6.16}$$

In this case, the first sum in Eq. (6.1) becomes

$$-\sum_{n=1}^{N}\frac{\phi_{n+1}\phi_n}{\epsilon^2}=-\frac{N(Aa)^2}{\epsilon^2}\left[\frac{1}{m}Z(\beta\epsilon)\mathrm{ds}(\beta\epsilon)-\mathrm{cn}(\beta\epsilon)\right]. \tag{6.17}$$

It is worth noting that all these sums are independent of the constant k_1 and hence c. As argued above, since the total energy E given by Eq. (6.1) is conserved (and hence time-independent) and since t and c always appear together in the combination k_1 in Eq. (6.1), it follows that the second sum in Eq. (6.1) must also be independent of k_1 and hence c. We thus have shown that the PN barrier is zero for the three periodic solutions and hence also for the localized solutions (which are obtained from them in the limit $m=1$).

In fact, it is easy to show that for all the three cases, the second sum (apart from a trivial k_1-independent constant) in Eq. (6.1) is given by

$$\sum_{n=1}^{N}\ln\left[1+\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)}\mathrm{dn}^2(n\beta\epsilon+k_1)\right]. \tag{6.18}$$

In particular, for the solution (6.12) the second sum is given by

$$B\sum_{n=1}^{N}\left\{\ln[\mathrm{cn}^2(\beta\epsilon)]+\ln\left[1+\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)}\mathrm{dn}^2(n\beta\epsilon+k_1)\right]\right\}. \tag{6.19}$$

On the other hand, for the solution (6.14), the second sum is given by

$$B\sum_{n=1}^{N}\ln\left[1+\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)}\mathrm{dn}^2(n\beta\epsilon+k_1)\right]. \tag{6.20}$$

Finally, for the solution (6.16), the second sum is given by

$$B\sum_{n=1}^{N}\left\{\ln\frac{\mathrm{cn}^2(\beta\epsilon)}{\mathrm{dn}^2(\beta\epsilon)}+\ln\left[1+\frac{\mathrm{sn}^2(\beta\epsilon)}{\mathrm{cn}^2(\beta\epsilon)}\mathrm{dn}^2(n\beta\epsilon+k_1)\right]\right\}. \tag{6.21}$$

It is worth remarking at this point that by following the above arguments, it is easily shown that even in the AL model [3] the PN barrier is zero for the periodic dn and cn solutions. In particular, since in that case the energy is essentially given by the first sum in Eq. (6.1), hence using Eqs. (6.15) and (6.17) it follows that indeed for both dn and cn solutions [2] there is no PN barrier in the AL model. Additional comments on this topic can be found in Ref. [24].

In general, we do not know how to write the sum in Eq. (6.18) in a closed form. However, for $m=1$ and $N\to\infty$, the sum of logarithms in Eq. (6.3) can be carried out explicitly, and the energy of the solitary wave can be given in a closed form.

We proceed as follows: One can show that, for $N\to\infty$, the following identity can be derived from the AL equation (see, for instance, Ref. [29]):

$$\sum_{n=-\infty}^{\infty}\ln\{1+\sinh^2(\rho)\mathrm{sech}^2[\rho(n-x)]\}=\frac{2}{\rho}. \tag{6.22}$$

Then, for kinklike solutions and $m=1$, we have

$$\begin{aligned}\ln(1-\alpha\phi_n^2)&=\ln[1-\tanh^2(\beta\epsilon)\tanh^2(n\beta\epsilon+k_1)]\\&=\ln\mathrm{sech}^2(\beta\epsilon)\\&\quad+\ln[1+\sinh^2(\beta\epsilon)\mathrm{sech}^2(n\beta\epsilon+k_1)].\end{aligned}$$

Thus, in the limit when $N\to\infty$ and $m=1$, the sum of logarithms in Eq. (6.3) becomes

$$\frac{2}{\beta\epsilon}+\sum_n\ln\mathrm{sech}^2(\beta\epsilon). \tag{6.23}$$

Similarly, in the case of the two pulse solutions, dn and cn are identical for $m=1$, and we can write

$$\ln(1-\alpha\phi_n^2)=\ln[1+\sinh^2(\beta\epsilon)\mathrm{sech}^2(n\beta\epsilon+k_1)].$$

In the limit when $N\to\infty$, the sum of logarithms in Eq. (6.3) for the two pulse solutions, dn and cn, for $m=1$, is simply equal to $2/(\beta\epsilon)$.

For the case of the kink solution, for $m=1$, B as given by Eq. (6.6) simplifies to

$$B=-\frac{\mathrm{sech}^2(\beta\epsilon)}{\alpha\epsilon^2}, \tag{6.24}$$

whereas for the case of the pulse solutions, at $m=1$, B simplifies to

$$B=-\frac{\cosh(\beta\epsilon)}{\alpha\epsilon^2}. \tag{6.25}$$

VII. CONCLUSIONS

In this paper, we have shown how to modify the naive discretization of $\lambda\phi^4$ field theory so that the discrete theory is a Hamiltonian dynamical system containing both static and moving solitary waves. We have found three different periodic elliptic solutions. To obtain time-dependent solitary wave solutions that were derivable from a Hamiltonian system which had the correct continuum limit requires a potential $V[\phi_n]$ that depends on the velocity of the solitary wave. The need for a velocity-dependent potential for this problem arises from a desire for the system to be both Hamiltonian and allowing moving solitary wave solutions. The dependence on the velocity vanishes as the lattice spacing goes to zero.

In the static case, we have studied the stability of both kinklike and pulselike solutions, and have found different qualitative behavior of the lowest eigenvalue of the stability matrix in the two cases. For typical values of the model parameters, in the case of kinklike solutions, we found that stability requires the number of sites, N, to be larger than a minimum value, while for pulselike solutions stability is achieved for arbitrary values of N. The magnitude of the lowest eigenvalues increases with N in the kinklike case, and decreases with N for pulselike solutions. The lattice spacing, ϵ, is not an independent parameter and always decreases with N.

We also determined the energy of the solitary wave in the three cases. Using the Hamiltonian structure, we were able to argue that the PN barrier [18,19] for all solitary waves is zero. As an additional result, we explicitly showed that for the two elliptic solutions (dn and cn) [2] of the integrable Ablowitz-Ladik model [3] the PN barrier is zero—as one would expect.

The single solitary wave solutions were found to be stable and when we scattered two such single-kink ($m=1$) solitary waves there were two different behaviors. For pulses, the pulse-antipulse solution leads to scattering with a flip and a little radiation (phonons). For kinklike solutions we found a breatherlike behavior [32] during the collision. However, we have *not* found *exact* two-solitary-wave or breather solutions, which would help clarify the integrable nature of this system.

The results presented here are useful for structural phase transitions [16,28] and possibly for certain field theoretic contexts [17]. Our results also hold promise for appropriate discretizations of other discrete nonlinear soliton-bearing equations [1,2]. Possible extension to discrete integrable models in 2+1 dimensions, e.g., Kadomtsev-Petviashvili (KP) hierarchies [34], would be especially desirable. Extension to time-discrete integrable models [35] is another interesting possibility.

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